ICERM

Lattice point counting and homogeneous dynamics

Part II

Lattice points in semi simple groups

G - semi-simple group

$\Gamma \leq G$ lattice (irreducible)

$B_T \leq G$ nice growing sets

Goal: Estimate $\# \Gamma \cap B_T$

Example

$G = SL_2(\mathbb{R})$ $\quad \Gamma = SL_2(\mathbb{Z})$

$B_T = \{ g \in G \mid \|g\| \leq T \}$

$\|g\|^2 = tr(g^t g) = a^2 + b^2 + c^2 + d^2$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad Vol(B_T) = \pi T^2$

Note: Identifying $SL_2(\mathbb{R}) / SO(2) = \mathbb{H}$

$g \mapsto g i$, $\quad B_T i = \{ z \in \mathbb{H} \mid d(z, i) < R_T \}$

$2\cosh(R_T) = T^2$

Geometric approach

cover each $\gamma \in \Gamma \cap B_T$ by fundamental domains: γF_Γ

Problems

1) For $\Gamma = SL_2(\mathbb{Z})$
$\Gamma \backslash G$ is not compact

2) for Γ co-compact
say $F_\Gamma \subseteq B_c$ for $c > 1$
$\gamma \in B_T$, $g \in F_\Gamma \Rightarrow \gamma g \in B_{cT}$
but
$Vol(B_{cT} \setminus B_T) \approx Vol(B_T)$

[Compare to $G = \mathbb{R}^n$ where
$B_T + v \subseteq B_{T+\varepsilon}$]

Use dynamics

Main tool: Mixing

Thm [Howe-Moore]

G s.s gp, $\Gamma \leq G$ irr lattice

$\Psi, \Phi \in L^2(\Gamma \backslash G)$

$$\int_{\Gamma \backslash G} \Psi(xh)\Phi(x)\,dx \xrightarrow{h \to \infty} \frac{(\int \Psi)(\int \Phi)}{\mathrm{Vol}(\Gamma \backslash G)}$$

Consequences:

1) For $U \subseteq G$ open translate Uh equidistributes in $\Gamma \backslash G$ as $h \to \infty$

2) For h_t one parameter group for a.e. $x \in \Gamma \backslash G$ orbit $\{xh_t \mid |t| \leq T\}$ equidistributes in $\Gamma \backslash G$

Counting via equidistribution

Let U_δ small neighborhood of I

Let $\varphi_\delta(g)$ supported on U_δ

and $$\Phi_\delta(x) = \sum_{\gamma \in \Gamma} \varphi_\delta(\gamma g) \in L^2(\Gamma \backslash G)$$

(Assume $U_\delta \subseteq F_\Gamma$)

Compute $$\int_{B_T} \Phi_\delta(g)\, dg$$

in two ways

I) $$\int_{B_T} \Phi_\delta(g)\, dg = \sum_{\gamma \in \Gamma} \int_G \chi_{B_T}(g)\, \varphi_\delta(\gamma g)\, dg$$

$$= \sum_{\gamma \in \Gamma} \int_G \chi_{B_T}(\gamma g)\, \varphi_\delta(g)\, dg$$

Regularity assumption:

Let $B_T^{\pm} = B_{T(1\pm\delta)}$

Assume: $\forall g \in B_T,\ h \in U_\delta$

$gh \in B_T^{+}$

With this assumption

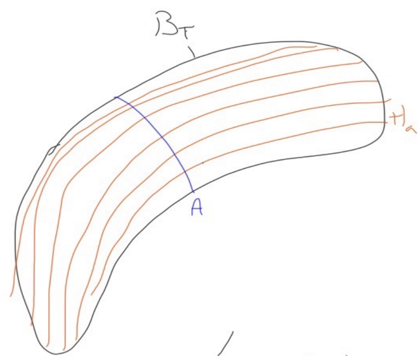
$$|B_T^{-} \cap \Gamma| \leq \int_{B_T} \Phi_\delta(g)\, dg \leq |B_T^{+} \cap \Gamma|$$

or

$$\int_{B_T^{-}} \Phi_\delta(g)\, dg \leq |\Gamma \cap B_T| \leq \int_{B_T^{+}} \Phi_\delta(g)\, dg$$

II) use equidistribution

Assume B_T disintegrates as union of long fibers each is equidistributed in ΓG



$$\int_{B_T} \Phi(g)\, dg = \int_A \int_{H_a} \Phi(h)\, dh\, da$$

$$\int_{H_a} \phi(h)\, dh \sim \frac{\int \phi(h)}{\mathrm{vol}(\Gamma G)} \cdot \int_{H_a} dh$$

Apply this to $G = SL_2(\mathbb{R})$ and B_T - norm balls

Coordinates

$g = k_\theta a_t k_{\theta'}$ $\quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$ $\quad 0 \leq t < \infty$

$0 \leq \theta \leq 2\pi$

Haar measure: $dg = \sinh(t)\, dt\, d\theta \frac{d\theta'}{2\pi}$

Norm: $\|g\|^2 = 2\cosh(t)$

$B_T = \{ k_\theta a_t k_{\theta'} \mid 0 \leq t \leq R_T \}$

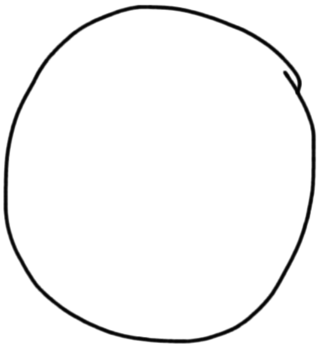
$2\cosh(R_T) = T^2$

$\mathrm{Vol}(B_T) = \pi T^2$

Two option for fibers:

1) Fix θ $\quad \{k_\theta a_t \mid 0 \le t \le R_T\}$

2) Fix t $\quad \{k_\theta a_t \mid 0 \le \theta \le 2\pi\}$



we use 2nd option and show

Thm: For $\psi \in C(\Gamma\backslash G)$

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(k_\theta a_t)\,d\theta \xrightarrow{t\to\infty} \frac{\int_{\Gamma\backslash G} \psi(g)\,dg}{\mathrm{Vol}(\Gamma\backslash G)}$$

Cor: $|\Gamma \cap B_T| \sim \dfrac{\mathrm{Vol}(B_T)}{\mathrm{Vol}(\Gamma\backslash G)}$

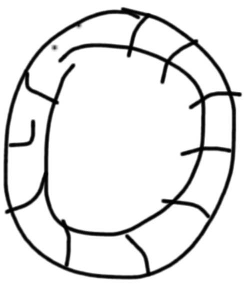
Note: what we expected from geometric argument

Proof of equidistribution

Idea: Use mixing: For open U, Ua_t equidistributes

Problem: Set $\{k_\theta \mid 0 \le \theta \le 2\pi\}$ not open

Solution: Thicken this set



but only thicken in directions that contract

Coordinates

$g = k_\theta a_s n_x \qquad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$dg = e^s \, ds \, d\theta \, dx$

Key observation

$k_\theta a_s n_x a_t = k_\theta a_t a_s n_{xe^{-t}}$

so if s, x small, $a_s n_x \in U_\delta$

then we have

$k_\theta a_s n_x a_t = k_\theta a_t h$ for $h \in U_\delta$

In this case

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(k_\theta a_t)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \varphi(k_\theta a_s n_x a_t)\,d\theta$$

$$+ o(1)$$

as $\delta \to 0$

Let $\varphi_\delta(k_\theta a_s n_x) = \varphi_\delta(a_s n_x)$

mean one and supported

in KU_δ

Writing $g = k_\theta a_s n_x$ we get

$$\int_G \Psi(g a_t)\,\varphi_\delta(g)\,dg = \frac{1}{2\pi}\int_0^{2\pi} \Psi(k_\theta a_t)\,d\theta + o(1)$$

on other hand

$$\int_G \Psi(g a_t)\,\varphi_\delta(g)\,dg = \int_{\Gamma\backslash G} \Psi(g a_t)\,\Phi_\delta(g)\,dg$$

$$\xrightarrow{t\to\infty} \frac{\left(\int_{\Gamma\backslash G}\Psi\right)}{\mathrm{vol}(\Gamma\backslash G)}$$

Taking $t \to \infty$ and $\delta \to 0$ concludes the proof.

Notes:

1) Mixing can be made effective so also equidistribution and counting estimates

2) Simple modification of proof shows

$$\frac{1}{I}\int_I \psi(k_a a_t)\,da \longrightarrow \frac{\int \psi\, dg}{vol(\Gamma\backslash G)}$$

(exe)

can use this to count in sectors

3) The same equidistribution result can be used to estimate $|\Gamma \cap \tilde{B}_T|$ with

$$\tilde{B}_T = \{ k_\theta a_t n_x \mid t \leq 2\log(T),\ |x| \leq \tfrac{1}{2} \}$$

exe: Action of G on $\mathbb{R}^2$

$g \mapsto g\begin{pmatrix}1\\0\end{pmatrix}$

$$\tilde{B}_T\begin{pmatrix}1\\0\end{pmatrix} = \{ v \in \mathbb{R}^2 \mid \|v\| \leq T \}$$

what counting result to we get in $\mathbb{R}^2$?